A Vector Electrical Network for Multi-Way Graph Partition

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Abstract
A system and method has been developed for performing multi-way graph partition. The partition is performed using a non-linear analog electrical network model. In this model, electrical potential and current are vectors. The electrical network consists of a set of non-linear resistors that correspond to the edges in the graph and have a current-limiting characteristic that is related to the edge strength. This algorithm applies to graphs which are undirected, have positive and real-valued edge weights and have at least two "label" nodes. Possible applications in image segmentation and depth perception are described. Other suggested applications of the algorithm are in non-rigid image registration and image-noise removal. (**September 24, 2017**)

Keywords
Multi-way graph cut — Flow Network — Image Segmentation — Depth Perception

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1. Introduction

A novel method is presented for the partition of graphs. The method applies to graphs that have the form of undirected flow networks and allows for multi-way partitioning. The flow network is an undirected weighted graph, G is defined as a set of nodes or vertices V, a set of edges E and a positive weight, w is associated with each edge:

$$G = (V, E, w) \tag{1}$$

where each edge, $\{p,q\}$ is defined as a pair of nodes $p,q \in V$. Additionally, a subset of the nodes, L are "label" nodes.

A graph cut, C is defined as a subset of the edges in the graph such that when those edges are removed from the graph the remaining graph consists of $|L|$ connected components and each connected component contains one node from the set L

A minimum-weight graph cut or minimum cut is a graph cut that has a minimum weight of all the possible graph cuts:

$$C_{min} = \underset{C\in\chi}{\operatorname{argmin}} \sum_{\{p,q\}\in C} w_{pq} \tag{2}$$

where χ is the set of all possible graph cuts.

An exact solution for the graph cut can be obtained for the case with binary labels [1], [2]. These methods are based on establishing a flow through the graph. All flows are allowed that conserve flow at each node and, for each edge in the graph are less than the weight of the edge. Another approach is to establish flow through the network. In this case, the flow is determined by resistors that represent each edge in the graph. This includes the models of Frisch [3] and Christiano *et al* [4]. A competitive algorithm for obtaining the graph cut has arisen based on the electrical model [9]. A novel methodology is presented here that generalizes the method of Yim [9] to apply to multi-way graph cuts. The method also extends the method of Grady and Funka-Lea for multi-label image segmentation [5].

2. The Multi-Way Graph Cut

An electrical network will obtain a state in which the voltages can be interpreted as the partition of the graph. This electrical network is related to the graph as follows. The electrical network is a network of resistors with each resistor representing one of the edges in the graph with the same connectivity as the corresponding edges in the graph. The electrical behavior of the resistors is non-linear and is given by the relation:

$$\mathbf{i}_{pq} = \frac{w_{pq}}{1+\|\mathbf{x}_p - \mathbf{x}_q\|}(\mathbf{x}_p - \mathbf{x}_q) \tag{3}$$

where $\mathbf{i}$ and $\mathbf{x}$ are vectors with $|L|$ components. The vector $\mathbf{i}$ is analogous to electrical current and the vector $\mathbf{x}$ is analogous to electrical voltage.

Each of the components of the current and voltage vectors are associated with a label in the graph. The relationship between labels and the voltage and current vectors is given by a map θ :

$$\theta : L \rightarrow \{\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_{|L|)}\} \tag{4}$$

where $e_1, e_2..$ are unit vectors in $\mathbb{R}^{|L|}$

Input to the electrical network is applied at the nodes in the electrical network corresponding to the nodes in the graph

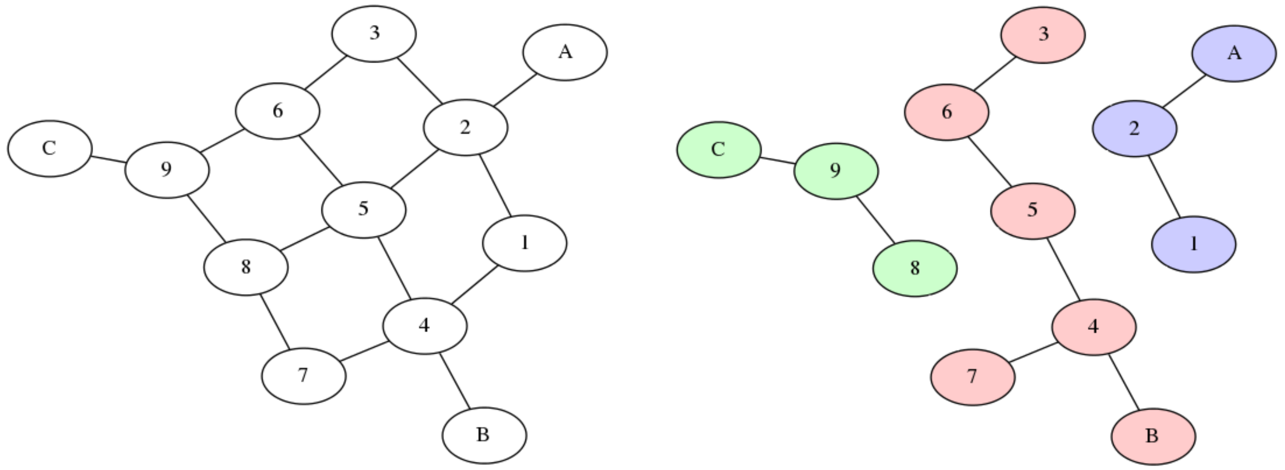

Figure 1. The multi-way graph cut. Edges are removed from the graph such that there is one connected component for each label node. The label nodes are $\{A,B,C\}$

in the set L. A vector-valued voltage is applied to each of the input nodes with the vector direction given by θ :

$$\mathbf{x}_l = v_{in}\theta(l) \tag{5}$$

where v_{in} is a large number and $l \in L$

Constraints are used to obtain a system of equations for the voltage. At each node, $p \in V \setminus L$ the net electrical current is constrained to be zero.

$$\sum_{q \in N_p} \mathbf{i}_{pq} = 0 \tag{6}$$

where N_p is the set of nodes in the electrical network that are directly connected to the node p

The systems of equations from (5) and (6) can be written in matrix form:

$$\mathbf{A}(\mathbf{X})\mathbf{X} = \mathbf{B} \tag{7}$$

where $\mathbf{A}(\mathbf{X})\mathbf{X}$ represents a factorization of the non-constant terms in the system of equations. $\mathbf{X}$ is a $|V| \times |L|$ matrix. Each row of $\mathbf{X}$ represents the unknown voltage vector at one of the nodes in electrical network, $\mathbf{x}_p$. $\mathbf{A}(\mathbf{X})$ is a $|V| \times |V|$ matrix and $\mathbf{B}$ is a $|V| \times |L|$ matrix and represents the constant terms in the system of equations.

The system of equations is solved using the fixed point method:

$$\mathbf{A}(\widetilde{\mathbf{X}}_k)\widetilde{\mathbf{X}}_{k+1} = \mathbf{B} \tag{8}$$

where $\widetilde{\mathbf{X}}$ is an approximate solution and k is the iteration. The fixed point solution is initialized with the zero matrix $\widetilde{\mathbf{X}}_0 = \mathbf{0}$

The voltage solution can be interpreted as a labelling of the nodes in the graph by comparing the relative magnitudes of the components of the voltage at each node. The label corresponding to the voltage component with the largest magnitude is assigned to the node:

3. Applications

3.1 Image Segmentation

Earlier work in the field of image segmentation has found a role for the graph cut [6],[7]. This method was extended by applying the proposed multi-way graph cut to image segmentation. In this study, the multi-way graph cut was applied to the segmentation of an image of the heart using computed tomography. A graph was constructed from the image:

$$G_{CT} = (V_{CT}, E_{CT}, w_{CT}) \tag{9}$$

where $V_{CT} = P_{CT} \cup L_{CT}$ where P_{CT} is the set of pixels in the image and L_{CT} is a set of 10 labels. Each node in the image has a value, f assigned to it. For pixel nodes, the value of f is the image intensity of the corresponding pixel. For label nodes, the value of f is the class intensity of the label. The class intensity is a value in the set $\{-100, -50, 0, ...350\}$. For each edge, e_{pq}, the difference in the intensity function f is given by a function δ_{pq}:

$$\delta_{pq} = |f(p) - f(q)| \tag{10}$$

The set of edges in the graph includes edges that produce a regularization effect, E_{CT^N} and edges that relate the pixel nodes to the labels E_{CT^L}:

$$E_{CT} = E^N_{CT} \cup E^L_{CT} \tag{11}$$

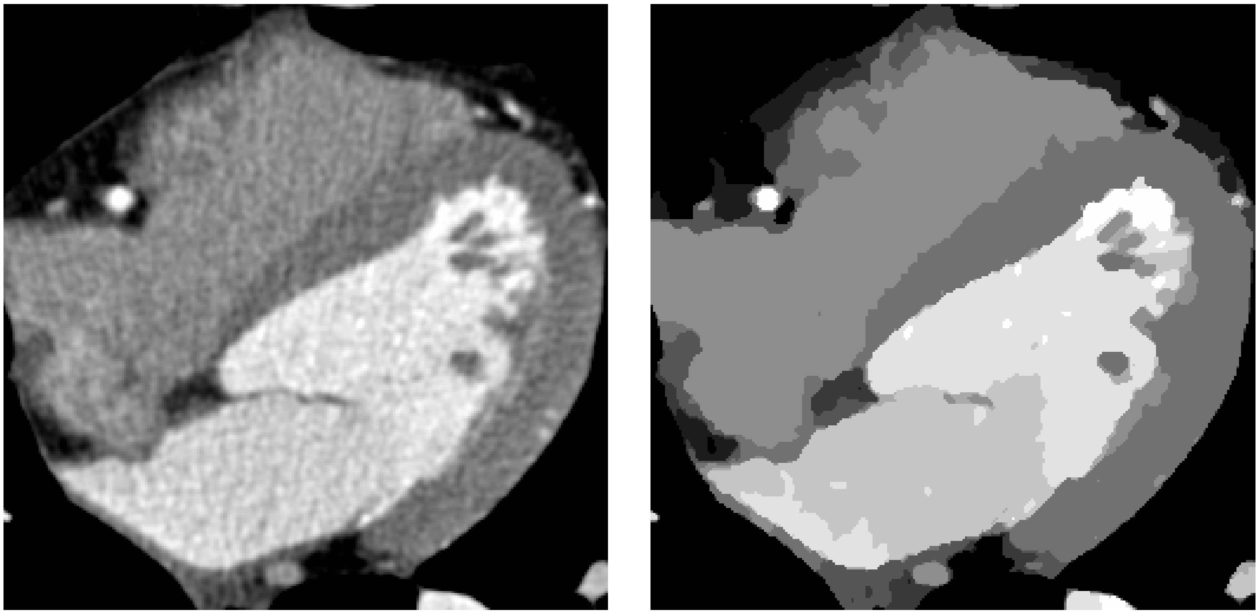

Figure 2. Image segmentation using the multi-way graph cut.

E_{CT^N} includes edges between all pairs of adjacent pixels in the image while E_{CT^L} includes edges between all pixel nodes in the image and all label nodes.

The weight of the edges is given by the function:

$$w_{CT}(e_{pq}) = \begin{cases} \frac{2.0}{1.0+\delta_{pq}{}^2}, \text{ if } e_{pq} \in E_{CT}^N \\ \frac{1.0}{1.0+\delta_{pq}{}^2}, \text{ if } e_{pq} \in E_{CT}^L \end{cases} \tag{12}$$

For construction of the graph, edges were included based on the 8-neighbor adjacency. The input voltage magnitude was $v_{in} = 10^6$. A solution was obtained for 10 fixed-point iterations. A direct solver was used at each fixed-point iteration. The algorithm was implemented in Python with the Numpy, Scipy, and Networkx modules. The segmentation result is shown in figure 2.

3.2 Depth Perception

The multi-way graph cut can be applied to depth perception from stereo photography [8]. This work is extended by the use of the proposed multi-way graph cut for performing the graph cut. The methodology is applied to a stereo photograph pair obtained from the Middlebury College Stereo Database. The stereo photograph pair was cropped for use with this algorthm.

In this application, each label represents the distance of a point in the photograph from the camera. A graph, G_{stereo} is constructed from a pair of photographs:

$$G_{stereo} = (V_{stereo}, E_{stereo}, w_{stereo}) \tag{13}$$

The graph, G_{stereo} has the same structure as the graph used for image segmentation. The set of nodes in the graph V_{stereo} are the pixels in the photograph from the left camera and the set of labels L_{stereo}. Edges are included between nodes of adjacent pixels and between each pixel and all of the label nodes:

$$E_{stereo} = E_{stereo}^N \cup E_{stereo}^L \tag{14}$$

where the function f_{left} is the color of a given pixel in the left photograph as a vector of the red, green and blue channels. For the edges in $^L_{stereo}$, the weights are given by:

$$w_{stereo}(e_{pq}) = \begin{cases} S(f_{left}(p), f_{left}(q)), \text{ if } e_{pq} \in E_{stereo}^N \\ S(f_{left}(p), f_{right}(t(e_{pq}))), \text{ if } e_{pq} \in E_{stereo}^L \end{cases} \tag{15}$$

where the function f_{left} and f_{right} are vectors representing the RGB channels of the color in the left and right photographs, respectively. $t(e_{pq})$ is a map from an edge to a pixel in the right photograph. The location of the pixel in the right photograph is the position of the pixel p shifted to the left by an amount that depends on the label node q.

S is defined as:

$$S(x,y) = 0.1 + \exp\left(\frac{-\|x-y\|^2}{100}\right) \tag{16}$$

The depth perception method was applied using 20 labels that represented apparent shifts between the left and right photographs of $\{0,1,....19\}$ in pixel units. Solution of the graph cut for depth perception was otherwise the same as for image segmentation. Results for depth perception using the multi-way graph cut are shown in figure 3.

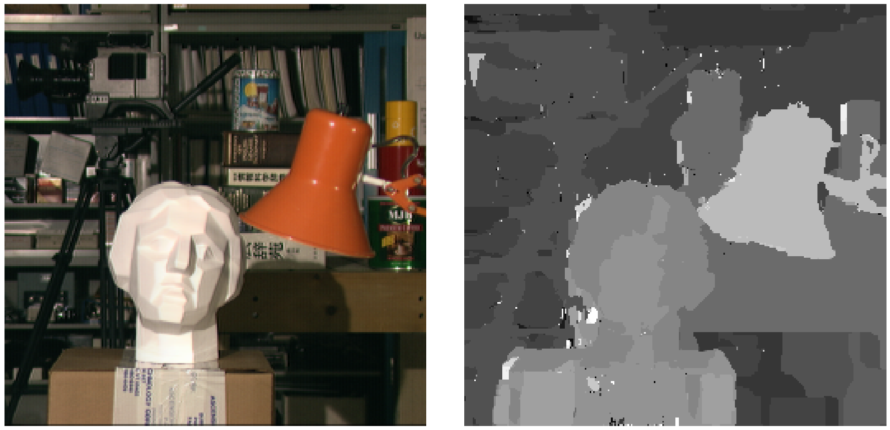

Figure 3. Depth perception using the multi-way graph cut.

4. Discussion

An elegant and promising method has been developed for performing the multi-way graph cut. The structure of the algorithm is comparable to the method in [9] and should be amenable to parallel implementation.

References

[1] Ford, L. R. and Fulkerson, D. R., ”Maximal flow through a network,” *Canadian Journal of Mathematics*, volume 8, pages 399-404, 1956.

[2] Goldberg, A. V. and Tarjan, R. E., ”A new approach to the maximum flow problem,” *Proceedings of the eighteenth annual ACM symposium on Theory of computing - STOC ’86*, pages 136-146, 1986

[3] Frisch, I. T., ”On electrical analogs for flow networks”, *Proceedings of IEEE*, volume = 57:2, pages 209-210, 1969

[4] Christiano, I. T. and Kelner J. A. and Madry, A. M. and Spielman, D. and Teng S. H., ”Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs,”, *STOC ’11, Proceedings of the 43rd Annual Symposium on Theory of Computing* pages 273-282, year 2010

[5] Grady, L. and FunkaLea, G., ”Multilabel image segmentation for medical applications based on graph theoretic electrical potentials,” *Computer Vision and Mathematical Methods in Medical and Biomedical Image Analysis*, volume = 3117, pages 230-245, 2004

[6] Ishikawa, H. and Geiger, D., ”Segmentation by grouping junctions,” *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, pages 125-131, 1998

[7] Boykov, Y. and Veksler, O. and Zabih, R., ”Markov random fields with efficient approximations,” *Proceedings of IEEE Computer Society Conference on Computer Vision and Pattern Recognition, 1998*, pages 648-655, 1998

[8] Boykov, Y. and Veksler, O., and Zabih, R., ”Fast approximate energy minimization via graph cuts,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, volume 23:11, pages 1222-1239, 2001

[9] Yim, P. J., ”US Patent 8929636, Method and system for image segmentation,” US 8929636, 2015